

# **The World of Hyperbolic Geometry**

Judy Maendel

*Mount Union College*

## The World of Hyperbolic Geometry

Well over 2,000 years ago, around 300 B.C., the Greek mathematician, Euclid, established several postulates which provide the foundation for what is now commonly referred to as Euclidean geometry. However, one of his postulates, the parallel postulate, caused some disturbance among mathematicians for a long period of time. The controversial Euclidean parallel postulate states that given a line,  $\ell$ , and any point,  $\mathbf{A}$ , not lying on  $\ell$ , there exists exactly one line passing through  $\mathbf{A}$  that is parallel to  $\ell$ .<sup>1</sup> The statement may seem intuitively obvious, as postulates should be; however, drawing two lines that are apparently parallel does not provide a solid argument that they actually are. This is because it is quite impossible to extend the drawn lines infinitely in both directions to demonstrate they, indeed, do not intersect. Even Euclid seemed to have recognized this problem in the way that he hesitantly and very sparingly used the postulate in his proofs.

Consequently, instead of accepting the statement as an axiom, many unfortunate souls labored endlessly, and unsuccessfully, over the course of 2,000 years to show that the remaining postulates could be used to actually prove the parallel postulate as a theorem (Smart 200-201). One such person, the Hungarian mathematician, Wolfgang (Farkas) Bolyai, lamented to his son about his own futile attempts saying, “I have traversed this bottomless night, which extinguished all light and joy of my life,” and advised him to “leave this science of parallels alone.” Fortunately, his son, Janos Bolyai (1802-1860), did not leave the matter alone and neither did the Russian mathematician, Nicholai Lobachevski (1793-1856). Both men simultaneously developed the same results independently of each other. Their approach to the parallel postulate was to reject it and carry out a proof by contradiction; however, no contradiction was reached. Instead, their attempts led them to recognize a new geometry, hyperbolic geometry. As a result, the confusion about parallels was eventually decided once and for all, for in this geometry a different parallel postulate was present:

**Hyperbolic Parallel Postulate:** Given a line,  $\ell$ , and any point,  $\mathbf{A}$ , not lying on  $\ell$ , there exist *at least two* lines through  $\mathbf{A}$  that are parallel to  $\ell$ .

However, this non-Euclidean geometry was too radical of a concept for that time in history. Bolyai and Lobachevski, unfortunately, did not receive the credit they deserved at the time their revolutionary accomplishments

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<sup>1</sup> This is actually not the original wording of Euclid’s fifth postulate, but it is nevertheless equivalent to it. It is also commonly referred to as Playfair’s axiom.

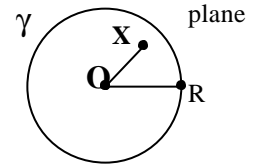
were published in 1832 and 1829, respectively, nor in their lifetime. In fact, it took 40 more years for the significance of their work to be realized by the mathematical world (Davis 101-103, Kay 394-395). Along with this realization, then, came the momentous question, “Which geometry is true, Euclidean or Hyperbolic (or any other of the non-Euclidean geometries)?” Both Bolyai and Lobachaveski held to the belief that this new geometry with its set of theorems was just as accurate as Euclid’s geometry at interpreting the world in which we live. In other words, if there are no inconsistencies in Euclid’s geometry then there are none in hyperbolic geometry, the converse being true as well. Although Euclidean geometry has not been proven to be *consistent* (void of contradictions), if it is assumed to be consistent, then the relative consistency of hyperbolic geometry can be proven.

Despite their remarkable work, neither, Bolyai or Lobachevski, achieved any proof of this, which involved “finding in each geometry a model of the other” (Coxeter 288-289). It was only late in the 1800’s that such a model was found. Indeed, two models representing the hyperbolic world were found, one of which will be discussed shortly. Subsequently, an interpretation of the undefined terms of geometry (“*points*,” “*lines*,” “*incidence*,” “*betweenness*,” and “*congruence*”) will be given and will then be shown to be a model in which the hyperbolic parallel property holds. In other words, assuming that all the axioms and propositions for Euclidean geometry are valid, it will be shown that the axioms of hyperbolic geometry hold under the given interpretation. Proving that the interpretation is a model of hyperbolic geometry implies its consistency relative to that of Euclidean geometry. That is, any inconsistency in Euclidean geometry will appear in hyperbolic geometry. Hence, with the assumption that Euclidean geometry is consistent, proving the Euclidean parallel postulate using the other postulates becomes an impossible task, for the postulate fails to hold in hyperbolic geometry. Ironically, then, if those well-intentioned mathematicians in their attempt to solidify Euclidean geometry had actually succeeded in proving Euclid’s parallel postulate, they would have completely weakened the geometry instead! (Greenberg 225-226)

The proof of relative consistency was presented in 1871 by the Beltrami-Klein model (or Klein model for short) which is due to the independent work of the Italian mathematician Eugenio Beltrami (1835-1900) and the German mathematician Felix Klein (1849-1925). This model was the first to prove the relative consistency of hyperbolic geometry (Davis 147). However, Klein’s model does not incorporate the usual Euclidean meaning of congruent angles and segments. This complicates the process of verifying the congruence axioms, although it can be done. Instead, a look at a model that improved upon Klein’s model will be given. This model is due to the French mathematician Henri Poincaré (1854-1912) who presented it in 1871. Poincaré’s disk model, like Klein’s

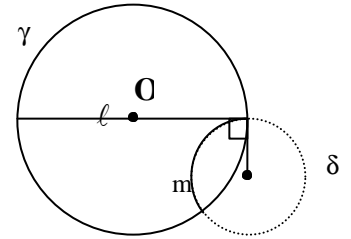
model, interprets congruent segments in an obscure fashion involving an elaborate definition of length. However, its interpretation of congruent angles holds the usual Euclidean meaning, making this model preferable to Klein's

(Davis 148-149). Hence, consider this interpretation where,  $\gamma$  is a circle in the Euclidean plane with center  $O$  and radius  $OR$ :



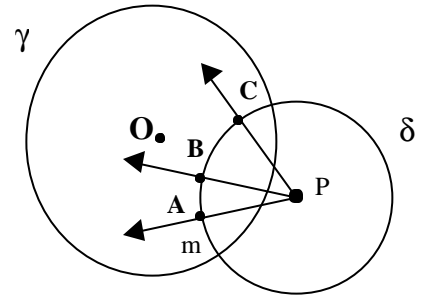
“points” - Points interior to a Euclidean circle  $\gamma$  ( $\{X: OX < OR\}$ )

“lines” - Open chords passing through center  $O$  of  $\gamma$  (open diameters  $\ell$  of  $\gamma$ )  
 Open arcs of circles orthogonal to  $\gamma$  (open circular arc  $m$  orthogonal to  $\gamma$ )  
 (Note: Either will be referred to as a Poincaré line or P-line and will be denoted as:  $A)(B.)$ )



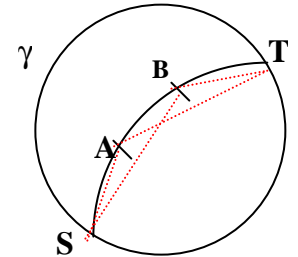
“incidence” - Euclidean sense of incidence

“between” - For open diameters  $\ell$  of  $\gamma$  the usual Euclidean interpretation holds.  
 For points  $A, B, C$  on an open circular arc  $m$  from an orthogonal circle  $\delta$  with center  $P$ , the point  $B$  is between the points  $A$  and  $C$  if ray  $\overline{PB}$  is between rays  $\overline{PA}$  and  $\overline{PC}$ .



The following definitions prepare the foundation for the interpretation of congruence:

The cross-ratio  $(AB, TS)$  is  $(\overline{AT} / \overline{BT}) (\overline{BS} / \overline{AS})$ , where  $A$  and  $B$  are points inside  $\gamma$ ,  $S$  and  $T$  are the ends of the P-line passing through  $A$  and  $B$ , and  $\overline{AB}$ , for example, is the Euclidean length of the Euclidean segment  $AB$ . Also,  $S$  and  $T$  are labeled so that  $A$  is between  $S$  and  $B$ .

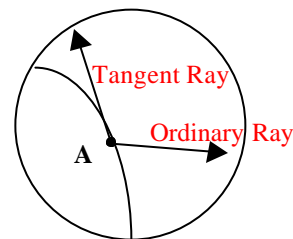
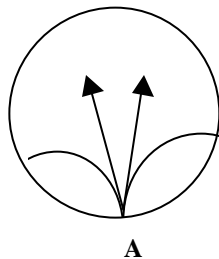


The Length of Segment AB is equal to  $\log(\text{cross-ratio}) = \log(AB, TS) = \log[(\overline{AT} / \overline{BT}) (\overline{BS} / \overline{AS})]$

Note:  $(AT/BT) > 1$  and  $(BS/AS) > 1$ , so  $(AT/BT)(BS/AS) > 1$ . Hence,  $\log[(AT/BT)(BS/AS)] > 0$  (Eves, 288). Poincaré length,  $d(AB)$  is defined, then, by  $d(AB) = |\log(AB, TS)|$ . (Greenberg, 248).

The measure of an angle is the radian measure of the angle formed by

- i. The tangent rays at a point  $A$  of two arcs intersecting at  $A$ , or
- ii. An ordinary ray and the tangent ray at a point  $A$  of an arc that intersects the ordinary ray at  $A$ .

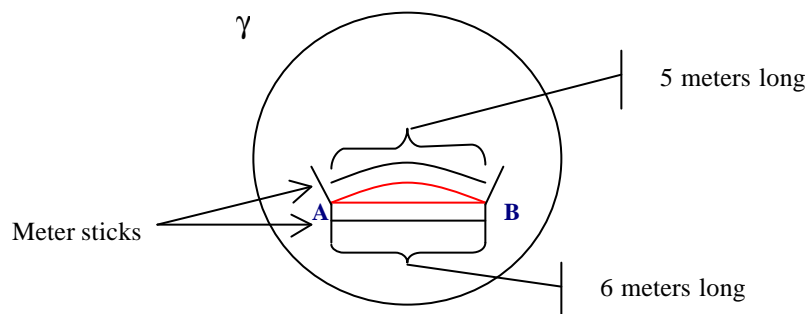


Given the preceding definitions, congruence is simply interpreted as follows:

- “congruence”
- Segments of equal length are congruent.
  - Angles of equal measure are congruent (Greenberg 232-233, 248; Eves 287-288).

Before diving into the proofs of the axioms in the Poincaré model, it is important to understand the characteristics Poincaré gave to his model representing the hyperbolic world. The following explanation, which will require a little imagination, should give adequate insight on the nature of the model. Suppose that inside a circle  $\gamma$ , with radius  $R$ , there is room enough for a substantial population of 2-dimensional people. You and I will be invisible giants looking in on their peculiar world. Oddly enough, the circle,  $\gamma$ , is filled with a strange gas that causes everything to shrink as it moves away from the center of  $\gamma$ . So, for instance, something  $X$  units in length at distance  $r$  from the center will be exactly  $X \cdot [1 - (r^2 / R^2)]$  units in length, where the half unit mark of an object is used to measure the object's distance,  $r$ , from the center of  $\gamma$ . For example, when the object is placed at the center,  $r = 0$ . When it is placed halfway to the rim,  $r = \frac{1}{2} R$ . When it is placed three-quarters of the way to the rim,  $r = \frac{3}{4} R$ , and so on. Thus, objects shrink yet stay in proportion to their surroundings. However, it so happens that these two-dimensional people with two-dimensional brains are not aware of this variation of linear dimensions they are experiencing. Consider a meter stick inside  $\gamma$ . “A man who at the center of  $\gamma$  is two meters tall, as measured by the meter-stick, will, after walking three-quarters of the way to the rim, still be two meters tall according to the very same stick.” Everything stayed in proportion, so he is not “aware that the stick, his body, his hat, his stride, as well as trees, cars, etc. are only .4375 as long as they used to be.”<sup>2</sup>

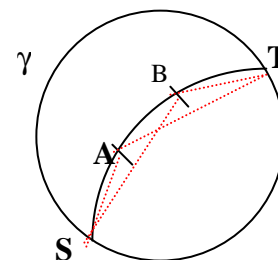
Another perplexing fact is that the gas inside  $\gamma$  makes light rays take the “shortest” path between two interior points, with the “shortest” path being measured by the two-dimensional people. We see the shortest path being a straight line, when the two points lie on a diameter of  $\gamma$ , and being a curve bulging toward the center, otherwise. This is “because meter-sticks get longer as they move in that direction.” See diagram:



<sup>2</sup> (length of the object (meter-stick) at distance  $r$  equals  $1 - (r^2 / R^2) = 1 - ((3/4)^2 R^2 / R^2) = 1 - (9/16) = 7/16 = 0.4375$  meters.

Eventually, these people in the flatland of  $\gamma$  (despite having two-dimensional brains) begin studying geometry, and not surprisingly, the geometry they create “reflects the universe as they perceive it.” Therefore, because objects shrink more and more the farther they get from the center of  $\gamma$ , they will never reach the rim of  $\gamma$ . For if they did, they would shrink to  $1 - (R^2/R^2) = 0$ , a most devastating effect, even on a two-dimensional mind! Thus, the world as they know it is infinite in all directions. (Trudeau 236-238)

However, it may seem unclear how lines in this model have infinite length because of the apparent boundary of the circle. Therefore, consider the segment  $AB$  of the P-line  $S(T)$ . The length of  $AB$  with respect to  $ST$  as defined earlier is  $\log[(\overline{AT} / \overline{BT}) (\overline{BS} / \overline{AS})]$ . The P-line  $A(B)$  is formed when  $A$  gets closer to  $S$  and  $B$  gets closer to  $T$ . When this occurs, the lengths of the segments  $AS$  and  $BT$  are obviously approaching zero. Hence, the equation for the length of  $AB$  tends to infinity.



*Circle Limit III* – M.C. Escher

To gain an enhanced visual perspective on this, consider a drawing by Dutch artist M.C. Escher (1898-1972). Escher’s Circle Limit drawings, which are each an elaboration of an earlier sketch by S.S.M. Coxeter, eloquently represent the hyperbolic world. Notice that the fish in *Circle Limit III* appear to be getting smaller the farther they are from the center of the circle. However, under the hyperbolic characteristics described in Poincaré’s flatland, they are actually all the same size. Furthermore, because an indefinite number of fish can theoretically be drawn along the P-lines (functioning here as fish’s

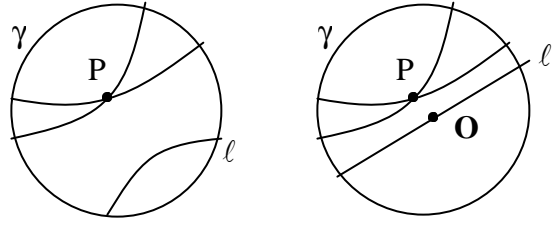
backbones), it becomes clear that P-lines have infinite length (Davis, 151-152).

Now that Poincaré’s peculiar little world has been thoroughly examined, and Escher has provided an aesthetically pleasing view of it, a look at the axioms of hyperbolic geometry for this interpretation will complete the picture.

First, the hyperbolic parallel property will be demonstrated prior to the validation of Poincaré’s axioms. (All the Euclidean axioms and propositions that are used in the proofs are provided in the appendix on page 10.)

**Hyperbolic Parallel Property:**

For every P-line  $\ell$  and every point P not on  $\ell$  in the interior of  $\gamma$ , there exist at least two P-lines through P, parallel to  $\ell$ .



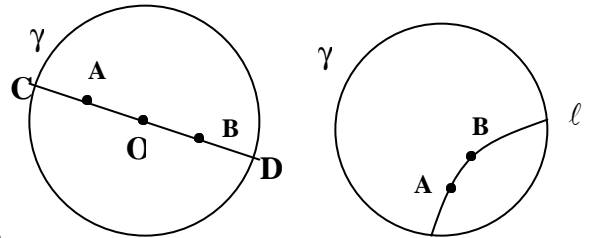
**Proof:**

Because the two P-lines passing through P do not intersect the P-line,  $\ell$ , inside the circle, they are parallel to  $\ell$ . It is of no concern, then, what the lines do outside the boundary of  $\gamma$ .

Now given are the incidence axioms, as they apply to Poincaré’s interpretation, and their proofs where necessary.

**Incidence Axiom 1 (Poincaré):**

Given any two distinct points A and B in the interior of circle  $\gamma$ , there exists a unique P-line,  $\ell$ , of  $\gamma$  such that A and B both lie on  $\ell$ .



**Proof:**

If A and B are two points interior to  $\gamma$  such that the Euclidean line  $\overline{AB}$  forms a diameter of  $\gamma$ , then the proof is as follows:

- |  |                  |
|--|------------------|
| 1. Let A and B be interior to $\gamma$ .                         | 1. Given         |
| 2. Let $\overline{AB}$ be the Euclidean line through A and B.    | 2. Axiom I-1     |
| 3. This line intersects $\gamma$ in two distinct points C and D. | 3. Proposition 4 |
| 4. A and B lie on the open chord C(D)                            | 4. Steps 2 and 3 |
| 5. This is the only open chord on which they both lie.           | 5. Axiom I-1     |

However, if A and B do not lie on a diameter, then from an elementary geometry theorem, it follows that there is a unique circle orthogonal to  $\gamma$  and passing through two given interior points A and B of  $\gamma$ . The arc of this unique circle is a P-line by definition. Thus there exists a unique P-line,  $\ell$ , of  $\gamma$  such that A and B both lie on  $\ell$  (Eves, 288).

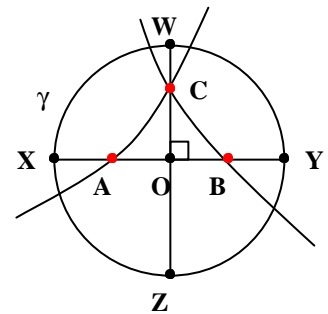
**Incidence Axiom 2 (Poincaré):**

For every P-line  $\ell$  there exist at least two distinct points incident with  $\ell$ . (Refer to above diagrams)

The proof is obvious

**Incidence Axiom 3 (Poincaré):**

There exist three distinct points interior to  $\gamma$  with the property that no P-line is incident with all three of them



**Proof:**

- |  |   |
|--|---|
| 1. Let $\gamma$ be a circle with center O.   | 1. Given  |
| 2. Let X and Y be two points lying on $\gamma$ such that the Euclidean line $\overline{XY}$ forms a diameter of $\gamma$ . | 2. Definition of Circle; Axiom I-1; Proposition 4 |
| 3. There exists points A and B, such that $X * A * O$ and $O * B * Y$ .  | 3. Axiom B-2                                      |
| 4. A and B are interior to $\gamma$ :  | 4. Given; Step 2; Axiom B-1; Proposition 2        |

$X * A * O$  and  $X * O * Y$  imply  $X * A * Y$  AND  
 $Y * B * O$  and  $Y * O * X$  imply  $Y * B * X$

5. Drop a perpendicular to  $\overline{XY}$  through  $O$ .
6. The perpendicular intersects  $\gamma$  in two distinct points, say  $W$  and  $Z$ .
7. There exists a point  $C$  such that  $W * C * O$ .
8.  $C$  is interior to  $\gamma$ :  
 $W * C * O$  and  $W * O * Z$  imply  $W * C * Z$
9. There exists a unique P-line through  $A$  and  $C$ .
10. There exists a unique P-line through  $C$  and  $B$ .
11. There exists three distinct points with the property that no P-line is incident with all three of them.

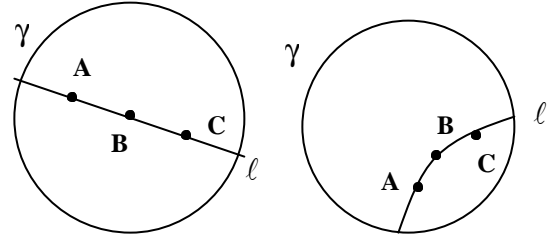
5. Proposition 3
6. Step 4; Proposition 4
7. Axiom B-2
8. Proposition 2
9. Poincaré's Axiom I-1
10. Poincaré's Axiom I-1
11. Steps 1-10

Next, the betweenness axioms will be handled.

**Betweenness Axiom 1 (Poincaré):**

If  $A, B,$  and  $C$  lie in the interior of  $\gamma$  such that  $A * B * C$ , then  $A, B,$  and  $C$  are three distinct points all lying on the same P-line, and  $C * B * A$ .

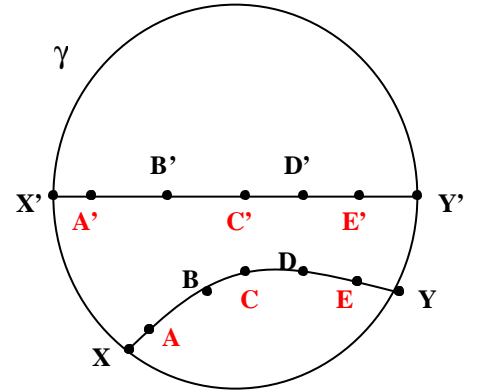
The proof is obvious.



**Betweenness Axiom 2 (Poincaré):**

Given any two distinct points  $B$  and  $D$ , in the interior of  $\gamma$  there exist point  $A, C,$  and  $E$  in  $\gamma$  lying on the P-line  $X)(Y$ , incident with  $B$  and  $D$  such that  $A * B * C, B * C * D,$  and  $B * D * E$ .

The proof is obvious.



**Betweenness Axiom 3 (Poincaré):**

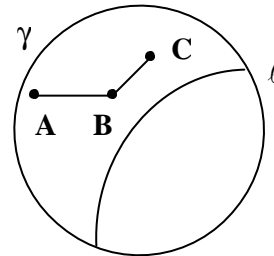
If  $A, B,$  and  $C$  are three distinct points in the interior of  $\gamma$  lying on the same P-line, then one and only one of the points is between the other two. That is, one and only one of the following can occur:  $A * B * C,$  or  $B * A * C,$  or  $B * C * A$ .

The proof is obvious.

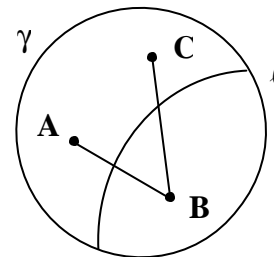
**Betweenness Axiom 4 (Poincaré):**

For every P-line,  $l,$  and for any three points  $A, B,$  and  $C$  not lying on  $l$ :

- (I) If  $A$  and  $B$  are on the same side of  $l$  and  $B$  and  $C$  are on the same side of  $l,$  then  $A$  and  $C$  are on the same side of  $l$ .



- (II) If  $A$  and  $B$  are on opposite sides of  $l$  and  $B$  and  $C$  are on opposite sides of  $l,$  then



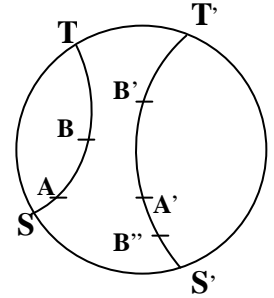
A and C are on the same side of  $\ell$ .

The proof is obvious.

Lastly, Poincaré’s congruence axioms are presented.

**Congruence Axiom 1 (Poincaré):**

If A and B are distinct points, and if A' is a point on the P-line S')(T', then there are two and only two points B' and B'' on S')(T' such that A'B'  $\cong$  AB and A'B''  $\cong$  AB; moreover, B' \* A' \* B''.



**Proof:**

Note that (A'T'/B'T')(B'S'/A'S') increases continuously from 1 to  $\infty$  as B' moves along S')(T' toward T'. Similarly, (A'S'/B''S')(B''T'/A'T') increases continuously from 1 to  $\infty$  as B'' moves along S')(T' toward S'. It follows that  $\overline{A'B'}$  and  $\overline{A'B''}$  increase continuously from 0 to  $\infty$  in the two cases. There are, therefore, unique positions of B' and B'' such that  $\overline{A'B'} = \overline{A'B''} = \overline{AB}$ . (Eves, 289)

**Congruence Axiom 2 (Poincaré):**

If AB  $\cong$  CD and AB  $\cong$  EF, then CD  $\cong$  EF. Moreover, every segment is congruent to itself.

The proof is obvious.

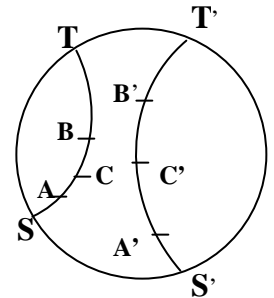
**Congruence Axiom 3 (Poincaré):**

If point C is between points A and B and point C' is between points A' and B', AC  $\cong$  A'C', and CB  $\cong$  C'B', then AB  $\cong$  A'B'.

**Proof:**

$$\begin{aligned} \overline{AB} &= \log[(\overline{AT}/\overline{BT})(\overline{BS}/\overline{AS})] \\ &= \log[(\overline{AT}/\overline{CT})(\overline{CS}/\overline{AS})(\overline{CT}/\overline{BT})(\overline{BS}/\overline{CS})] \\ &= \log[(\overline{AT}/\overline{CT})(\overline{CS}/\overline{AS})] + \log[(\overline{CT}/\overline{BT})(\overline{BS}/\overline{CS})] \\ &= \overline{AC} + \overline{CB}. \end{aligned}$$

Similarly,  $\overline{A'B'} = \overline{A'C'} + \overline{C'B'}$ .



However, since  $\overline{AC} = \overline{A'C'}$  and  $\overline{CB} = \overline{C'B'}$  from the given,  $\overline{AB} = \overline{A'B'}$ . Thus, AB  $\cong$  A'B' (Eves 290).

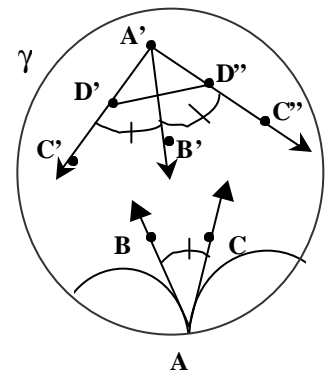
**Congruence Axiom 4 (Poincaré):**

If  $\angle BAC$  is an angle whose sides do not lie in the same line and if A' and B' are two distinct points, then there are two and only two distinct rays,  $\overrightarrow{A'C'}$  and  $\overrightarrow{A'C''}$ , such that  $\angle B'A'C' \cong \angle BAC$  and  $\angle B'A'C'' \cong \angle BAC$ ; moreover, if D' is any point on the ray  $\overrightarrow{A'C'}$ , and D'' is any point on the ray  $\overrightarrow{A'C''}$ , then D'D'' intersects the line determined by A' and B'.

**Proof:**

The stated congruence axiom is a direct result of the following theorem:

There is a unique “circle” orthogonal to a given circle  $\gamma$  and tangent to a given line  $\ell$  at an ordinary point of  $\ell$  not on  $\gamma$  (Eves 290).



**Congruence Axiom 5 (Poincaré):**

If  $\angle A \cong \angle B$  and  $\angle A \cong \angle C$ , then  $\angle B \cong \angle C$ . Moreover, every angle is congruent to itself.

The proof is obvious

**Congruence Axiom 6 (SAS) (Poincaré):**

If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent. (Figure 2)

Because the verification of this axiom deals with concepts involving the inversion of a circle, it is beyond the scope of this paper to give a complete explanatory proof. For those who would like to traverse this long but not bottomless path, a complete proof of this axiom can be found in College Geometry by Howard Eves (pp.290-292). With that, the last of the axioms is verified, making Poincaré's interpretation an accurate model of hyperbolic geometry!

Both the Klein model and Poincaré's disk model provided proof of the relative consistency of hyperbolic geometry, securing its validity with that of Euclidean geometry. The implications of this are fascinating. Just as the strange 2-dimensional people of Poincaré's world were not aware of the proportional downsizing of objects moving away from the center, we may have the same limited scope of our world.

To understand this better, consider the following definition of angle sum: The *angle sum* of triangle ABC is  $(\angle A)^\circ + (\angle B)^\circ + (\angle C)^\circ$ . In Euclidean geometry (where we assume the Euclidean parallel postulate) the angle sum of a triangle is exactly  $180^\circ$ , a fact all should recall from a 9<sup>th</sup> or 10<sup>th</sup> grade geometry class. However, in neutral geometry, where none of the various parallel postulates are assumed to hold, it can only be proven that the angle sum is *less than or equal to*  $180^\circ$ . Furthermore, in hyperbolic geometry (where we obviously assume the hyperbolic parallel postulate), it can be verified that a triangle has angle sum *strictly less than*  $180^\circ$ . Now, the geometry that most believe our world to be based on is that of Euclid's, where triangles have an angle sum of exactly  $180^\circ$  (Greenberg 130). Because this is the geometry we adhere to, it is quite impossible in the world, as we know it, to find a triangle for which this is not true. Nevertheless, Lobachevsky, one of the original non-Euclidean thinkers, did attempt to find such a triangle. He reasoned that in order to accomplish his goal, gargantuan triangles needed to be used. For instance, he considered the rather large "triangle formed by the star Sirius and two diametrically opposed position of the earth." (See Figure 3)

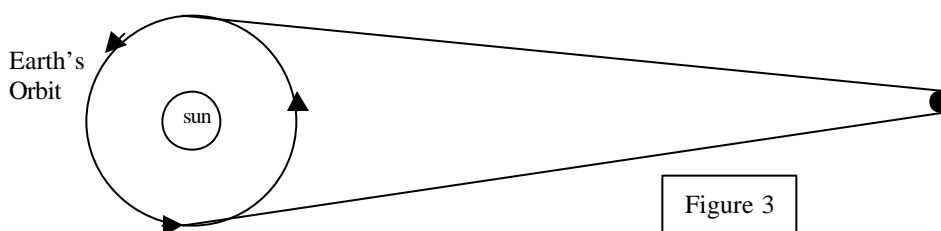


Figure 3

Alas, his results were inconclusive and, as mentioned before, did not go over well with the mathematical world, who knew and embraced only Euclid's geometry (Davis 105). However, having considered Poincaré's model, we conclude that Euclid's geometry cannot be singled out as being the *only* logical geometry just because it seems to represent our world well. This is just as Poincaré's flatland people should not assume hyperbolic geometry to be the only valid geometry because that is all *they* know. Imagine if we, who are outsiders of their world, knocked on their door and told them that just outside their circle was a world in which a different geometry held. The impact on their little 2-dimensional brains would be, at the least, mind boggling! Likewise, just because we "outsiders" are 3-dimensional does not mean that just outside our "circle" there is not a different geometry that holds.

Thus, it should be quite evident why the discovery of these new geometries was so upsetting to many individuals who ascertained Euclid's geometry to be absolute truth. Perhaps, as new technology is developed to reach farther beyond the world, as we know it, it will become known whether the non-Euclidean geometries come into the picture. For the present, an open mind about the matter is wise, even when things appear absolutely obvious (Meyer 113-114).

### Appendix

Proposition 1: For every line  $\ell$  there is at least one point not lying on it.

Proposition 2:  $A * B * C$  and  $A * C * D$  if and only if  $B * C * D$  and  $A * B * D$ .

Proposition 3: For every line  $\ell$  and every point  $P$  there exists a unique line through  $P$  perpendicular to  $\ell$ .

Proposition 4: If line  $\ell$  passes through a point  $A$  inside circle  $\gamma$  then  $\ell$  intersects  $\gamma$  in two points.

Axiom I-1: For every point  $P$  and for every point  $Q$  not equal to  $P$  there exists a unique line  $\ell$  incident with  $P$  and  $Q$ .

Axiom B-1: If  $A * B * C$ , then  $A$ ,  $B$ , and  $C$  are three distinct points all lying on the same line, and  $C * B * A$ .

Axiom B -2: Given any two distinct points  $B$  and  $D$ , there exist points  $A$ ,  $C$ , and  $E$  lying on  $BD$  such that  $A * B * D$ ,  $B * C * D$ , and  $B * D * E$ .

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