Eleventh Annual ECC Undergraduate Mathematics Competition, April 5, 2008

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

* * Time control: three hours * *

1. Adding logs.

Given that p and q are positive real numbers satisfying $\log p + \log q = \log(p+q)$, express p as a function of q. What is the domain of this function?

2. Unknown coefficients.

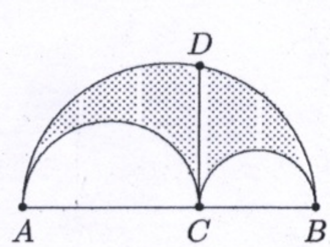
The polynomial $P(x) = 2x^3 + Ax^2 + Bx - 3$ is exactly divisible by $(x-1)$, and leaves a remainder of 9 when divided by $(x+2)$. Find A and B.

3. A common root.

For what values of b do the equations $2008x^2 + bx + 8002 = 0$ and $8002x^2 + bx + 2008 = 0$ have a common root?

4. Area of the arbelos.

Point C lies on the diameter AB of a semicircle as shown at the right. Semicircles with diameters AC and CB are drawn. Point D on the large semicircle is chosen so that CD is perpendicular to AB. The region (shaded) above the two smaller semicircles and below the large one is called an *arbelos*. Show that the area of the arbelos is equal to the area of the circle with diameter CD.



5. Solve for x.

Find all real roots of the equation

$$\sqrt{2008}x^{\log_{2008} x} = x^2.$$

6. Probability of three heads.

A certain coin is biased in such a way that when it is tossed five times, the probability of getting heads exactly once is the same as that of getting heads exactly twice (and this probability is not 0). Find the probability of getting heads exactly three times in five tosses.

7. Limit of a function.

Evaluate

$$\lim_{x \to -\infty} (3x + \sqrt{9x^2 - 7x + 2}),$$

if it exists, and justify your answer.

8. Minimum value.

Find the minimum value of

$$\frac{x}{3y} + \frac{6y}{z} + \frac{4z}{x} + 4$$

for positive numbers x, y, z.

9. How many positive integer pairs?

Determine, with proof, how many ordered pairs (x, y) of positive integers there are such that

$$\frac{xy}{x+y} = 2008.$$

10. An integral.

Evaluate the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^3 x dx}{\sin^3 x + \cos^3 x}.$$

(Hint: Try the substitution $y = \frac{\pi}{2} - x$.)