Solutions, First Annual ECC Undergraduate Mathematics Competition, April 4, 1998

1. The necklace.

Its value is \$2900, as we now show. Let x be the value of the middle diamond (in dollars). Then the values of its neighbors to one side are $x-100$, $x-200$, $\cdots$, $x-1000$, and those to the other side, $x-150$, $x-300$, $\cdots$, $x-1500$. By pairing each one on one side with the corresponding one on the other side we may express the total value as

$$\begin{aligned} 47150 &= x + (2x-250) + (2x-500) + \cdots + (2x-2500) \\ &= 21x - 250(1+2+3+\cdots+10) \\ &= 21x - (250)(55) \\ &= 21x - 13750, \end{aligned}$$

so $21x = 47150 + 13750 = 60900$, and $x = 2900$.

2. The millionth digit.

The 10^6-th digit is 1. It is the initial 1 in the integer 185185. To see this, note that there are 9 1-digit integers, 90 2-digit integers, 900 3-digit integers, etc. At the end of the last k-digit integer we have a total of

$$9 + 2\cdot 90 + 3\cdot 900 + \cdots + k\cdot 9\cdot 10^{k-1}$$

digits. With $k=6$, this is more than 10^6, but with $k=5$ it is 488889 digits. We need $511111 = 85185\cdot 6 + 1$ more digits, so we pass the first 85185 6-digit numbers, and the first digit of the 85186-th 6-digit number is our quarry. This number is $100000+85185 = 185185$, and the desired digit is the initial 1 in this integer.

3. Sum the series.

We will show that the sum is $6e$. Denote the numerator of the general term by $P(n)$. Rewrite $P(n)$ algebraically as follows:

$$P(n) = 2n^3 + n^2 - 4n - 2 = 2n(n-1)(n-2) + 7n(n-1) - n - 2.$$

Since rearrangement of convergent positive term series leaves sums unchanged, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{P(n)}{n!} &= 2\sum_{n=0}^{\infty} \frac{n(n-1)(n-2)}{n!} + 7\sum_{n=0}^{\infty} \frac{n(n-1)}{n!} - \sum_{n=0}^{\infty} \frac{n}{n!} - 2\sum_{n=0}^{\infty} \frac{1}{n!} \\ &= 2\sum_{n=3}^{\infty} \frac{1}{(n-3)!} + 7\sum_{n=2}^{\infty} \frac{1}{(n-2)!} - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - 2\sum_{n=0}^{\infty} \frac{1}{n!} \\ &= 2e + 7e - e - 2e = 6e.\end{aligned}$$

4. Max and min.

The maximum value is 603.8, at $x = -5$. The minimum value is $-10\sqrt{2}$, at $x = \sqrt{2}$. Here is the proof: $|x+1|$ is the distance from x to -1, and this is between 2 and 4 precisely when x is in $[-5, -3] \cup [1, 3]$.

When $x < 0$,

$$f(x) = -4x^3 - 21x + 6x^{-1}$$

and

$$f'(x) = -12x^2 - 21 - 6x^{-2},$$

which is negative, so $f(x)$ decreases throughout the interval $[-5, -3]$. The extreme values on this interval are $f(-5) = 603.8$ and $f(-3) = 169$.

When $x > 0$,

$$f(x) = 4x^3 - 21x + 6x^{-1}$$

and

$$\begin{aligned}f'(x) &= 12x^2 - 21 - 6x^{-2} \\ &= \frac{3}{x^2}(4x^4 - 7x^2 - 2) \\ &= \frac{3}{x^2}(4x^2 + 1)(x^2 - 2).\end{aligned}$$

In the interval $[1, 3]$, the derivative is 0 only at $x = \sqrt{2}$, so the candidates for extrema here are 1, $\sqrt{2}$ and 3. The corresponding function values are $f(1) = -11$, $f(\sqrt{2}) = -10\sqrt{2}$ and $f(3) = 47$. From this it is clear that the maximum and minimum values are as asserted.

5. An inequality.

Because of the symmetry in a and b we may assume WLOG that $a \leq b$. Then $a^2 \leq ab \leq b^2$, so that

$$a^2 \leq c^2 \leq b^2.$$

Thus $(a-c) \leq 0$ and $(b-c) \geq 0$, and the desired result follows.

6. Determinants.

(a) By adding the elements of the last row of M to each of the other rows, we obtain the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since this matrix has determinant $2^4 = 16$, we conclude that $\det(M) = 16$.

(b) If we add the elements of the first row of K to each of the other four rows, each of these four rows is divisible by 2, so we may factor out a 2 from each. Consequently, $\det(K)$ is a multiple of 16, and the desired result follows.

7. Not a square.

Write $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

$$P(4) - P(1) = a_n(4^n - 1) + a_{n-1}(4^{n-1} - 1) + \cdots + a_1(4-1).$$

Since $u^k - 1$ is divisible by $u - 1$ for every integer $k \geq 1$, $4^k - 1$ is divisible by 3 for every such k, and therefore $P(4) \equiv P(1) \equiv 2 \pmod 3$. But 2 is not a square mod 3, so $P(4)$ is not a square mod 3, and therefore is not a square integer.

8. Solutions in integers.

Upon clearing the equation of fractions we obtain $(a+b)^2 = nab$. Thus a/b must satisfy the equation $x^2 + (2-n)x + 1 = 0$. The only possible rational solutions are 1 and -1. Since $a+b$ cannot be 0, we must have $a = b$, which in turn requires $n = 4$. Conversely, $a = b \neq 0$ and $n = 4$ satisfies. Thus the solutions are

$$\{(a, a, 4) : a \neq 0\}.$$

9. How many regions?

The maximum number is $100^2 + 1 = 10001$. We will show that with n such parabolas, the maximum number of regions is $n^2 + 1$.

We first show that no more than $n^2 + 1$ are possible. The graphs of two such funtions can have at most two intersections, since the equation

$$a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$$

has at most two real roots. Therefore, when $k-1$ such graphs have been drawn, the next (i.e., the k-th) one can have at most $2k-2$ crossings of these $k-1$ graphs, and so can pass through (and hence divide) at most $2k-1$ regions, creating at most $2k-1$ new regions. (Actually, the region at the far left may be the same as that at the far right, but a new curve cutting through both the far left portion and the far right portion creates 2 new regions just as if it were passing through two old regions.) With $k = 1$ there are two regions, and the second graph can add at most 3 new regions. Thus, when the n-th graph is drawn, the number of regions is at most

$$2 + 3 + 5 + \cdots + (2n-1) = n^2 + 1.$$

We now show that $n^2 + 1$ are always possible. Let the first function be $y = x^2$ and the second $y = x^2/2 + 1$. It is easy to check that this crosses the first in two points and creates three new regions, for a total of $5 = 2^2 + 1$. Suppose now that k graphs $y = a_ix^2 + c_i$ have been drawn, with each of $a_1, \ldots, a_k$ positive, dividing the plane into $k^2 + 1$ regions. Let c_{k+1} be 1+ (the maximum of the y-coordinates of the intersection points in these k graphs) and $a_{k+1} = (1/2)\min\{a_1, \ldots, a_k\}$. Then the graph of

$$y = a_{k+1}x^2 + c_{k+1}$$

has vertex at $(0, c_{k+1})$, above all earlier intersections and all earlier vertices, and for $|x|$ sufficiently large,

$$a_{k+1}x^2 + c_{k+1} < \min_{1 \leq j \leq k}\{a_jx^2 + c_j\},$$

so all k earlier graphs have been crossed twice, creating $2k+1$ new regions above this last parabola. The total number of regions is then $k^2 + 1 + (2k+1) = (k+1)^2 + 1$. By induction, the claim follows.

10. A functional equation.

We show that the only such function is $f(x) = 1 + \frac{1}{x}$.
From (2) with $x = y$ we have

$$2f\left(\frac{1}{x}\right) = 1 + f\left(\frac{1}{2x}\right), \tag{3}$$

and putting $t = \frac{1}{2x}$ we have

$$2f(2t) = 1 + f(t) \quad \text{for all } t \neq 0. \tag{4}$$

Then for all $x \neq 0$,

$$\begin{aligned} x(1 + f(x)) &= 2xf(2x) && \text{by (4)} \\ &= f\left(\frac{1}{2x}\right) && \text{by (1)} \\ &= 2f\left(\frac{1}{x}\right) - 1 && \text{by (3)} \\ &= 2xf(x) - 1 && \text{by (1);} \end{aligned}$$

i.e.,

$$x(1 + f(x)) = 2xf(x) - 1. \tag{5}$$

Upon dividing by x in (5) we have $1 + f(x) = 2f(x) - \frac{1}{x}$, from which we obtain

$$f(x) = 1 + \frac{1}{x}$$

as the only candidate.
Substitution verifies that this function does satisfy (1) and (2).